Received: *March 7, 2001*, Accepted: *April 5, 2001*

Branes on group manifolds, gluon condensates, and twisted K-theory

Stefan Fredenhagen and Volker Schomerus

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut
Am Mühlenberg 1, D–14476 Golm, Germany
E-mail: stefan@aei.mpg.de, vschomer@aei.mpg.de

Abstract: In this work we study the dynamics of branes on group manifolds G deep in the stringy regime. After giving a brief overview of the various branes that can be constructed within the boundary conformal field theory approach, we analyze in detail the condensation processes that occur on stacks of such branes. At large volume our discussion is based on certain effective gauge theories on non-commutative 'fuzzy' spaces. Using the 'absorption of the boundary spin'-principle which was formulated by Affleck and Ludwig in their work on the Kondo model, we extrapolate the brane dynamics into the stringy regime. For supersymmetric theories, the resulting condensation processes turn out to be consistent with the existence of certain conserved charges taking values in some non-trivial discrete abelian groups. We obtain strong constraints on these charge groups for $G = \mathrm{SU}(N)$. The results may be compared with a recent proposal of Bouwknegt and Mathai according to which charge groups on curved spaces X (with a non-vanishing NSNS 3-form field strength H) are given by the twisted K-groups $K^*_H(X)$.

Keywords: D-branes, Conformal Field Models in String Theory, Non-Commutative Geometry.

JHEP04(2001)007

Contents

1. Introduction

During the last years, the study of branes and their dynamics has lead to many new insights into string and M-theory. Much of this study was done in the large volume regime where geometric techniques provide reliable information. The extrapolation into the stringy regime usually requires new methods from boundary conformal field theory, in particular when the bulk supersymmetry is not maximal. The analysis of strings and branes on group manifolds gives us a good handle on such issues. The large symmetry of group manifolds G makes string theory on G rather tractable while, on the other hand, group manifolds display many interesting new features that do not appear in flat spaces. Most importantly, their non-vanishing curvature along with the string equations of motion imply that they carry a non-vanishing NSNS 3-form field strength H. Moreover, models of strings and branes on group manifolds are used as a starting point in perturbative string constructions for many other backgrounds.

Our focus in this work is on bound state formation of branes and on finding appropriate conserved quantities (charges) that encode the essential features of the brane dynamics. D-branes in a background X may be characterized by their ability to carry RR-charges [1]. The latter are assigned to arbitrary configurations of branes, stable and unstable, and they are conserved during all dynamical processes. In the world-sheet description, D-brane configurations correspond to boundary conditions for some 2D conformal field theory and their condensation is induced by relevant (or marginally relevant) boundary operators. The infra-red (IR) fixed point of the associated renormalization group (RG) trajectory provides the world-sheet theory for the decay product that is reached after the condensation has occurred. In this framework, the conserved RR-charges are simply RG-invariants.

By construction, the brane charges take values in some discrete abelian group. Obviously, the latter contains a lot of information about the brane dynamics (i.e. RG trajectories) and hence it is rather difficult to find. On the other hand, there are many constructions in mathematics that assign discrete abelian groups to a background geometry X. These include the de Rham cohomology and various different K-theories. Very naively one might think that RR-charges take values in de Rham cohomology groups $H^*(X,\mathbb{Z})$ as they are associated with the n-form fields of supergravity theories. It is by now well known that this naive expectation is incorrect and that K-groups provide a much more realistic candidate for the group of RR-charges (see e.g. [2] and the more recent developments [3]–[8] that were initiated mainly by [9]).

There exist various different K-theories that one uses depending on the string theory under consideration. For type IIA/B theory in a background X, the relevant groups are given by the usual $K^*(X)$, provided that X carries a vanishing NSNS 3-form H. In dealing with the general case $H \in H^3(X,\mathbb{Z})$, Bouwknegt and Mathai proposed to employ the twisted K-groups $K^*_H(X)$. The latter are defined as K-groups of an algebra whose elements are sections of some bundle on X taking values in compact operators. Morita invariance of algebraic K-theory implies that one recovers $K(X)$ for $H = 0$. When the H-field is torsion class, i.e. some integer multiple of it vanishes in $H^3(X,\mathbb{Z})$, the proposal of Bouwknegt and Mathai boils down to K-groups suggested in [3] (see also [11] for an extensive discussion).

Having all these different groups at our disposal, it is important to decide which one gives the right answer, i.e. leads to some RG-invariants in boundary conformal field theories. To begin with, this requires some background X for which the various groups are actually different. In finding such examples one needs to overcome de Rham's theorem which claims that the non-torsion parts of de-Rham cohomology and

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of the usual K-theory are isomorphic.[1]Hence, one has to work on string backgrounds for which the cohomology and K-groups possess torsion parts. Such examples are known (see e.g. [12]), but to study brane dynamics in such backgrounds still presents a challenge.

String theory on group manifolds provides an interesting class of examples for which the appropriate twisted K-theory is rather non-trivial while at the same time there exist powerful methods to study the dynamics of branes. Group manifolds G are curved so that by the string equations of motion they come equipped with a non-trivial H-field, known as the WZW 3-form. The latter is non-torsion and hence we expect the brane charges to take values in the twisted K-groups $K^*_H(G)$.

The simplest example is given by $G = \mathrm{SU}(2)$. It was shown in [13] that branes on SU(2) can wrap integer conjugacy classes. Their stability was also analyzed in [14, 15]. Generically, conjugacy classes of SU(2) are 2-spheres but there are two exceptional point-like classes provided by $\pm e$ where e is the group unit. The large volume analysis of [16] shows that an arbitrary spherical brane on SU(2) may be obtained as a bound state of some sufficiently large stack of point-like branes, similarly to the effect described in [17]. This implies that all branes carry a certain integer multiple of the charge of a single point-like brane. When the volume of $\mathrm{SU}(2) \cong S^3$ is finite, the condensation can lead us from a sufficiently large stack of point-like branes at e to a single point-like brane at $-e$ [18]. Hence, the authors of [18] concluded that the charge of a point-like brane does not take values in the integers but in some finite quotient thereof. We shall see that the result is in perfect agreement with the K-theoretic prediction. The investigations in [18] were motivated by [19] (see also [20] and the more recent work [21]).

Other groups admit branes which wrap more general 'twisted' conjugacy classes [22]. Their dynamics can still be analyzed even deep in the stringy regime and the analysis gives strong constraints on the possible charge groups. In particular, we shall see that they are all *finite* discrete abelian groups but our information will be much more detailed. For $G = \mathrm{SU}(N)$ we shall show that the charge group is of the form

$$C(\mathrm{SU}(N), K) = \mathbb{Z}_x \oplus \mathbb{Z}_y \oplus \cdots , \tag{1.1}$$

where $x = K/\gcd(K, \mathrm{lcm}(1, \ldots, N))$ and y is known to divide x when $N \leq 5$ or N odd. For even $N \geq 6$ our results on y are slightly weaker and we defer their precise formulation to section 4.3 below. The integer K is determined by the NSNS 3-form $H \in H^3(\mathrm{SU}(N), \mathbb{Z}) \cong \mathbb{Z}$. Branes wrapping ordinary conjugacy classes contribute the first summand to eq. (1.1) while the second comes with twisted branes. Further summands could come from other types of branes for which there is no conformal field theory construction known yet. For $G = \mathrm{SU}(2)$ the second summand does not

[1]A finitely generated abelian group C is of the form $C = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{x_1} \oplus \cdots \oplus \mathbb{Z}_{x_n}$. The subgroup $\mathrm{Tor}(C) := \mathbb{Z}_{x_1} \oplus \cdots \oplus \mathbb{Z}_{x_n} \subset C$ is called the *torsion* of C.

appear. In case of $G = \mathrm{SU}(3)$, the conformal field theory studies give $y = x$. In both cases, our answer is very likely to be complete and it agrees with the predictions from twisted K-theory.

The plan of this paper is as follows. In the next section we will review the theory of maximally symmetric branes on group manifolds. In particular, we shall provide a complete list of such branes and their associated open string spectra. Section 3 is devoted to the dynamics of branes on group manifolds. We start with a brief summary and generalization of the results obtained in [16] for the large volume regime. The main aim of the section is then to explain how condensation processes can be studied deep in the stringy regime and to present explicit results on bound states. This information is then used in section 4 to derive strong constraints on the charge groups. In section 5, finally, we make some remarks on the comparison with twisted K-theory. These will be rather preliminary, though, because there is not much known about the twisted K-groups beyond the examples of $G = \mathrm{SU}(2), \mathrm{SU}(3)$.

2. D-branes on group manifolds

This section is devoted to the description of maximally symmetric branes on group manifolds. Following [13, 22], we will begin with a brief review of their classical geometry. Then, in the second subsection, we shall present some basic results on the boundary conformal field theory of such branes.

2.1 The geometry of branes on group manifolds

Strings on the group manifold of a simple and simply connected group G are described by the WZW-model. Its action is evaluated on fields $g : \Sigma \mapsto G$ taking values in G and it involves one (integer) coupling constant k, which is known as the 'level'. For our purposes it is most convenient to think of k as controlling the size (in string units) of the background. Large values of k correspond to a large volume of the group manifold. When dealing with open strings at tree level, the 2-dimensional world sheet Σ is taken to be the upper half plane $\Sigma = \{z \in \mathbb{C} | \operatorname{Im} z \geq 0\}$.

Along the boundary of this world sheet we need to impose some boundary condition. Here we shall analyze boundary conditions that preserve the full bulk symmetry of the model, i.e. the affine algebra $\widehat{\mathcal{G}}_k$. These boundary conditions are formulated in terms of the chiral currents

$$J(z) = k\, g^{-1}(z,\bar{z})\partial g(z,\bar{z})\,, \qquad \bar{J}(\bar{z}) = -k\, \bar{\partial} g(z,\bar{z})\, g^{-1}(z,\bar{z})\,.$$

Note that J and $\bar{J}$ take values in the finite dimensional Lie algebra $\mathcal{G}$ of the group G. Along the real line we glue the holomorphic and the anti-holomorphic currents according to

$$J(z) = \Lambda \bar{J}(\bar{z}) \qquad \text{for all} \qquad z = \bar{z}\,, \tag{2.1}$$

where Λ is an appropriate automorphism of the current algebra $\widehat{\mathcal{G}}_k$ (see e.g. [23]). The choice of Λ is restricted by the requirement of conformal invariance which means that $T(z) = \bar{T}(\bar{z})$ all along the boundary. Here $T, \bar{T}$ are the non-vanishing components of the stress energy tensor. They can be obtained through the Sugawara construction, as usual.

The allowed automorphisms Λ of the affine Lie algebra $\widehat{\mathcal{G}}$ are easily classified. They are all of the form

$$\Lambda = \Omega \circ \mathrm{Ad}_g \qquad \text{for some} \quad g \in G\,. \tag{2.2}$$

Here, Ad_g denotes the adjoint action of the group element g on the current algebra $\widehat{\mathcal{G}}_k$. It is induced in the obvious way from the adjoint action of G on the finite dimensional Lie algebra $\mathcal{G}$. The automorphism Ω does not come from conjugation with some element g. More precisely, it is an outer automorphism of the current algebra. Such outer automorphisms $\Omega = \Omega_\omega$ come with symmetries ω of the Dynkin diagram of the finite dimensional Lie algebra $\mathcal{G}$. One may show that the choice of ω and $g \in G$ exhausts all possibilities for the gluing automorphism Λ in eq. (2.2) (see e.g. [24]).

Throughout this text, the groups $G = \mathrm{SU}(N)$ will serve as our main examples. Their Dynkin diagrams A_{N-1} possess the trivial symmetry $\omega = \mathrm{id}$ and one non-trivial involution ω when $N > 2$. We will not need explicit expressions for the associated maps Ω_ω, but the interested reader can find formulas e.g. in [25].

So far, our discussion of the possible types of gluing automorphisms Λ has been fairly abstract. But it is possible to associate some concrete geometry with each choice of Λ. This was initiated in [13] for $\omega = \mathrm{id}$ and extended to non-trivial symmetries $\omega \neq \mathrm{id}$ in [22] (see also [26, 27]).

Let us assume first that the element g in eq. (2.2) coincides with the group unit $g = e$. This means that $\Lambda = \Omega = \Omega_\omega$ is determined by ω alone. The diagram symmetry ω induces an (outer) automorphism $\omega_{\mathcal{G}}$ of the finite dimensional Lie algebra $\mathcal{G}$ through the unique correspondence between vertices of the Dynkin diagram and simple roots. After exponentiation, $\omega_{\mathcal{G}}$ furnishes an automorphism ω_G of the group G. One can show that the gluing conditions (2.1) force the string ends to stay on one of the following ω-twisted conjugacy classes

$$C_u^\omega \;:=\; \{\, hu\,\omega_G(h^{-1}) \mid h \in G \,\}\,.$$

The subsets $C_u^\omega \subset G$ are parametrized by equivalence classes of group elements u where the equivalence relation between two elements $u, v \in G$ is given by: $u \sim_\omega v$ iff $v \in C_u^\omega$. Note that this parameter space U^ω of equivalence classes is not a manifold, i.e. it contains singular points.

To describe the topology of C_u^ω and the parameter space U^ω (at least locally), we need some more notation. By construction, the action of $\omega_{\mathcal{G}}$ on $\mathcal{G}$ can be restricted to an action on the Cartan subalgebra $\mathcal{T}$. We shall denote the subspace of elements which are invariant under the action of $\omega_{\mathcal{G}}$ by $\mathcal{T}^\omega \subset \mathcal{T}$. Elements in $\mathcal{T}^\omega$ generate a torus $T^\omega \subset G$. One may show that the generic ω-twisted conjugacy class C_u^ω looks like the quotient G/T^ω. Hence, the dimension of the generic submanifolds C_u^ω is $\dim G - \dim T^\omega$ and the parameter space has dimension $\dim \mathcal{T}^\omega$ at all but finitely many points. In other words, there are $\dim \mathcal{T}^\omega$ directions transverse to a generic twisted conjugacy class. This implies that the branes associated with the trivial diagram automorphism $\omega = \mathrm{id}$ have the largest number of transverse directions. It is given by the rank of the Lie algebra.

As we shall see below, not all these submanifolds C_u^ω can be wrapped by branes on group manifolds. There exists some integrality requirement that can be understood in various ways, e.g. as quantization condition within a semiclassical analysis [13] of the brane's stability [14, 15]. This implies that there is only a finite set of allowed branes (if k is finite). The number of branes depends on the volume of the group measured in string units.

Let us become somewhat more explicit for $G = \mathrm{SU}(N)$. The simplest case is certainly $N = 2$ because there exists no non-trivial diagram automorphism ω. The conjugacy classes C_u^{id} are 2-spheres $S^2 \subset S^3 \cong \mathrm{SU}(2)$ for generic points u and they consist of a single point when $u = \pm e$ in the center of SU(2). More generally, the formulas $\dim \mathrm{SU}(N) = (N-1)(N+1)$ and $\mathrm{rank}\ \mathrm{SU}(N) = (N-1)$ show that the generic submanifolds C_u^{id} have dimension $\dim C_u^{\mathrm{id}} = (N-1)N$. In addition, there are N singular cases associated with elements u in the center $\mathbb{Z}_N \subset \mathrm{SU}(N)$. The corresponding submanifolds C_u^{id} are 0-dimensional. Note that all the submanifolds C_u^{id} are even dimensional. Similarly, the generic manifolds C_u^ω for the non-trivial diagram symmetry ω have dimension $\dim C_u^\omega = (N-1)(N+1/2)$ for odd N and $\dim C_u^\omega = N^2 - N/2 - 1$ whenever N is even. For some exceptional values of u, the dimension can be lower.

So far we restricted ourselves to $\Lambda = \Omega_\omega$ being a diagram automorphism. As we stated before, the general case is obtained by admitting an additional inner automorphism of the form Ad_g. Geometrically, the latter corresponds to rigid translations on the group induced from the left action of g on the group manifold (see e.g. [28]). The freedom of translating branes on G does not lead to any new charges and we shall not consider it any further, i.e. we shall assume $g = e$ in what follows.

2.2 Conformal field theory

The branes we considered in the previous subsection may be described through an exactly solvable conformal field theory. In particular, there exists a complete understanding of the open string spectra based on the work of Cardy [29] and of Birke, Fuchs and Schweigert [25].

We shall use $\Xi = (\alpha, \omega)$ to label the boundary conformal field theories. The label α is taken from some index set $\mathcal{J}_k^\omega$ depending on the choice of the diagram

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automorphism ω and on the level k. For the trivial diagram automorphism $\omega = \mathrm{id}$, the set $\mathcal{J}_k = \mathcal{J}_k^{\mathrm{id}}$ coincides with the set of primaries of the affine Kac-Moody algebra $\widehat{\mathcal{G}}_k$. As is well known, $\mathcal{J}_k$ is a certain subset of the set $\mathcal{J} = \mathcal{J}_\infty$ of equivalence classes of irreducible representations for the finite dimensional Lie algebra $\mathcal{G}$. The automorphism ω generates a map $\varpi_k : \mathcal{J}_k \to \mathcal{J}_k$. In fact, given an irreducible representation τ of $\mathcal{G}$, we can define another representation by composition $\tau \circ \omega_{\mathcal{G}}$. The class of $\tau \circ \omega_{\mathcal{G}}$ is independent of the choice of $\tau \in [\tau]$ and so we obtain a map $\varpi : \mathcal{J} \to \mathcal{J}$. The latter descends to $\mathcal{J}_k \subset \mathcal{J}$. A label $i \in \mathcal{J}_k$ is said to be (ω-)symmetric, if it is invariant under the action of ϖ, i.e. if $\varpi i = i$. The subset of symmetric labels will be denoted by $\mathcal{J}_k^\omega \subset \mathcal{J}_k$. According to the results of [29, 25], the labels α for branes associated with the diagram automorphism ω take values in the set $\mathcal{J}_k^\omega$.

These very formal constructions can be understood as follows: obviously we would like to think of α in $\Xi = (\alpha, \omega)$ as labeling the position of the brane transverse to the ω-twisted conjugacy classes. As we explained before, the transverse space is locally given by $\mathcal{T}^\omega$. This fits nicely with our description of the sets $\mathcal{J}_k^\omega$. In fact, by construction the labels $\alpha \in \mathcal{J}_k^\omega$ run through a set of points on some lattice of dimension $\dim \mathcal{T}^\omega$. When $\omega = \mathrm{id}$, this lattice coincides with the weight lattice of $\mathcal{G}$.

Our main goal here is to explain the open string spectra that come with these branes. For a pair of boundary labels $(\alpha, \omega), (\beta, \omega)$ associated with the same diagram automorphism ω, the partition function is of the form

$$Z^\omega_{\alpha\beta}(q) = \sum_{j \in \mathcal{J}_k} n^{\omega;\beta}_{j\alpha} \chi_j(q) \, . \tag{2.3}$$

Here, $\chi_j(q)$ denote the characters of the current algebra $\widehat{\mathcal{G}}_k$ and $\alpha, \beta \in \mathcal{J}_k^\omega$. Consistency requires the numbers $n^{\omega;\beta}_{j\alpha}$ to be non-negative integers.

There exists a very simply argument due to Behrend et al. [30] which shows that the numbers $n^{\omega;\beta}_{j\alpha}$ give rise to a representation of the fusion algebra of $\widehat{\mathcal{G}}_k$. This means that they obey the relations

$$\sum_{\beta \in \mathcal{J}_k^\omega} n^{\omega;\beta}_{i\alpha} n^{\omega;\gamma}_{j\beta} = \sum_{k \in \mathcal{J}_k} N_{ij}{}^k n^{\omega;\gamma}_{k\alpha} \, , \tag{2.4}$$

where N_{ij}^k are the fusion rules of the current algebra $\widehat{\mathcal{G}}_k$. The argument of [30] starts from a general ansatz for the boundary state assigned to (α, ω). Using world sheet duality, one can express the numbers n in terms of the coefficients of the boundary states and the modular matrix S for the current algebra $\widehat{\mathcal{G}}_k$. The general form of this expressions is then sufficient to check the relations (2.4) (see [30] for details).

An explicit construction for the numbers n is given in [31, 25]. Let ω be a diagram automorphism, as before. Then the numbers n^ω are of the form

$$n^{\omega;\beta}_{i\alpha} = \sum_{\lambda \in \mathcal{J}_k^\omega} \frac{S^{\omega\,*}_{\lambda\beta} S^\omega_{\lambda\alpha} S_{\lambda i}}{S_{\lambda 0}} \qquad \text{for} \qquad \alpha, \beta \in \mathcal{J}_k^\omega \quad \text{and} \quad i \in \mathcal{J}_k \, . \tag{2.5}$$

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The matrix S^ω is a unitary matrix with matrix elements $S^\omega_{\lambda\alpha}$ indexed by the ϖ-symmetric labels $\lambda, \alpha \in \mathcal{J}^\omega_k$, i.e. they obey $\varpi\lambda = \lambda$ and $\varpi\alpha = \alpha$. When $\omega = \mathrm{id}$, the matrix S_ω coincides with the usual S-matrix so that Verlinde's formula implies

$$n^{\mathrm{id};\beta}_{i\alpha} = N_{i\alpha}^{\ \ \beta} \qquad \text{for all} \qquad \alpha, \beta, i \in \mathcal{J}_k\,.$$

This reproduces Cardy's results on the boundary partition functions [29]. For non-trivial automorphism ω, the matrix S^ω describes modular transformations of twisted characters. Explicit formulas for S^ω exist (see e.g. [25]), but we will not need them here.

Some aspects of the formulas (2.3), (2.5) with $\alpha = \beta$ can be understood geometrically. Let us note first that the partition functions of all our boundary theories are obtained by summing characters of the $\widehat{\mathcal{G}}_k$ algebra. This reflects the fact that all the (twisted) conjugacy classes admit an obvious action of the Lie group G by (twisted) conjugation.

The spectrum of ordinary conjugacy classes can be explained in much more detail. For simplicity, we shall restrict to $G = \mathrm{SU}(2)$. In this case, generic conjugacy classes are 2-spheres and the space of functions thereon is spanned by spherical harmonics $Y^{j/2}_m$, $|m| \leq j/2$ and $j = 0, 2, 4, \ldots$[2] The space of spherical harmonics is precisely reproduced by ground states in the boundary theory (α, id) when we send α (and hence k) to infinity. For finite α, the angular momentum j is cut off at a finite value $j = \min(2\alpha, 2k{-}2\alpha) \leq 2\alpha$. This means that the brane's world-volume is 'fuzzy' since resolving small distances would require large angular momenta. The relation between branes on SU(2) and the familiar non- commutative fuzzy 2-spheres [32, 33] was fully analyzed in [34] and it provides the only known example of a open string non-commutative geometry beyond the familiar case of branes in flat space [35]–[37]. The analysis of [34] goes much beyond the study of partition functions as it employs detailed information on the operator product expansions of open string vertex operators based on [38]. Using the results in [22, 39] it is easy to generalize all these remarks on ordinary conjugacy classes to other groups (see also [32] for more details and explicit formulas on fuzzy conjugacy classes).

Twisted conjugacy classes are more difficult to understand. This is related to the fact that they are never 'small'. More precisely, it is not possible to fit a generic twisted conjugacy class into an arbitrarily small neighborhood of the group identity unless the twist ω is trivial. This implies that the spectrum of angular momenta in $Z^\omega_{\alpha\alpha}$ is not cut off before it reaches the obvious large momentum cut-off that is set by the volume of the group, i.e. by the level k. For large α (and large k) the ground states in the boundary theory $\Xi = (\alpha, \omega)$ span the space of functions on the generic twisted conjugacy classes C^ω_u [22]. The non-commutative geometry associated with twisted conjugacy classes with finite α, however, remains to be investigated.

[2] To be consistent with our treatment of $\mathrm{SU}(N)$ below, we use a convention in which the spin is labeled by integers rather than half-integers.

2.3 Supersymmetric WZW models

Throughout this paper we shall address supersymmetric WZW-models. The main effect on our considerations shows up in a shift of the level k. In fact, a supersymmetric SU(N) model at level $K = k + N$ describes strings moving on SU(N) with K units of NSNS-flux. The model contains currents J^a satisfying the relations of a level $k + N$ affine Kac-Moody algebra along with a multiplet of free fermionic fields ψ^a in the adjoint representation of $su(N)$. It is well known that one can introduce new *bosonic currents*

$$J^a_{\mathsf{b}} \ := \ J^a + \frac{i}{k} f^a{}_{bc} \psi^b \psi^c$$

which obey again the commutation relations of a current algebra but now the level is shifted to $K - N = k$. The fermionic fields ψ^a commute with the new currents. This means that the theory splits into a product of a level k WZW model and a theory of $\dim G$ free fermionic fields.

This split is consistent with the boundary conditions we study. We want to impose gluing conditions $J^a(z) = \Lambda \bar{J}^a(\bar{z})$ and $\psi^a(z) = \pm\Lambda\bar{\psi}^a(\bar{z})$ along the boundary $z = \bar{z}$. This implies $J^a_{\mathsf{b}}(z) = \Lambda \bar{J}^a_{\mathsf{b}}(\bar{z})$ since Λ is an automorphism of the Kac-Moody algebra so that, in particular, its action on the product of the fermionic fields is intertwined by the structure constants of the Lie algebra. Hence, boundary conditions of the level K model are described by boundary conditions for the free fermions and for a level k bosonic current. Since the fermionic and the bosonic sectors decouple, we can restrict our attention to the latter. But eventually our results should be interpreted in terms of the original supersymmetric background which carries $K = k + N$ units of NSNS-flux.

3. Brane dynamics on group manifolds

The central goal now is to understand the dynamics of branes on group manifolds deep in the stringy regime when the group manifolds become small in string units. As a starting point it is useful to consider condensation processes in a large volume expansion, i.e. an expansion in powers of $1/k$. There one can make reliable and complete statements on the renormalization group flows using rather elementary techniques. As usual, the stringy regime is much more difficult to attack so that we cannot claim to have a complete understanding of the renormalization group flows and fixed points for small group volumes. But fortunately, a large number of condensation processes have been looked at in the past. In fact, the questions we are considering are very closely related to the Kondo problem for which a lot of technology has been invented during the last decades. As was remarked in [16, 18], the results carry over to the study of gauge field condensates on group manifolds. Finally, the comparison with the large volume scenario suggests that the essential processes are captured by our analysis.

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3.1 Brane dynamics at large volume

In our discussion of the large volume dynamics we follow very closely the studies of [16]. We consider a stack of M identical and symmetry preserving branes of type $\Xi = (\omega, \alpha)$. This configuration preserves the full $\widehat{\mathcal{G}}_k$ chiral algebra and hence we find all the $\widehat{\mathcal{G}}_k$ currents among the boundary operators of the corresponding conformal field theory. The rest of the field content depends very much on the particular brane Ξ we consider. But in a supersymmetric theory after removal of the tachyonic modes, these additional fields will become more and more massive as we decrease the size of our group manifold.[3] Since we are ultimately interested in the stringy regime, this justifies to concentrate on the currents which are the only massless fields away from the $k \to \infty$ limit. In other words, we restrict to perturbations of the form

$$S_{\text{pert}} \;=\; \int_{\partial\Sigma} dx A_a J^a(x)\,, \tag{3.1}$$

where x is the coordinate on the boundary $\partial\Sigma$ of the world-sheet and $A_a, a = 1, \ldots, \dim G$ is a set of $M \times M$ Chan-Paton matrices. Adding a perturbation of this form will give the gauge fields and transverse scalars on the brane a constant vacuum expectation value A_a.

The rules of perturbative string theory relate the effective action for the fields A_a to the correlation functions of the boundary currents $J^a(x)$. But the latter are completely determined by the usual Ward identities and hence they are entirely independent of the boundary condition Ξ we have selected. This observation implies that the associated terms in the brane's effective action are universal, i.e. they can be computed once and for all without reference to the boundary condition we are looking at. The results of these computations can be copied from [16],

$$S_{M\Xi}(A) = \text{tr}\ \left(-\frac{1}{4}\,[\, A_a\,,\, A_b\,]\ [\, A^a\,,\, A^b\,] + \frac{i}{3}\, f^{abc}\, A_a\, [\, A_b\,,\, A_c\,] + \text{const} \right). \tag{3.2}$$

Here, f^{abc} are the structure constants of the Lie algebra $\mathcal{G}$ and tr is the trace on the space of $M \times M$-matrices. From eq. (3.2) we obtain the following equations of motion

$$\Big[\, A^a \ \ , [\, A_a\,,\, A_b\,] \;-\; i\, f_{abc}\, A^c \Big] = 0\,. \tag{3.3}$$

As we remarked before, neither the effective action nor the equations of motion depend on the brane Ξ. But they certainly depend on the number M of branes that we stack together through the size of the matrices A_a.

Now we have to study solutions of the equations (3.3). It turns out that there are basically two types of solutions. The first one is given by a set of $\dim G$ pairwise commuting $M \times M$ matrices A_a. It comes as a $M \cdot \dim G$ parameter family of solutions corresponding to the number of eigenvalues appearing in $\{A_a\}$. The same

[3] A large number of boundary fields becomes marginal in the limit $k \to \infty$.

kind of solutions appears also for branes in flat backgrounds and the interpretation is known from [40]. They describe individual rigid translations of the M branes on the group manifold. Since each brane's position is specified by $\dim G$ coordinates, the number of parameters matches nicely with the interpretation. Moving branes around in the background is a rather trivial operation so that we need not consider this type of solutions any further.

There exists a second type of solutions to eqs. (3.3) which is a lot more interesting. In fact, any M-dimensional representation of the Lie algebra $\mathcal{G}$ can be used to solve the equations of motion. At least for untwisted branes, i.e. for $\omega = \mathrm{id}$, the interpretation of these solutions was found in [16]. Let us describe the answer for general $\Xi = (\alpha, \mathrm{id})$ and an irreducible M-dimensional representation σ of $\mathcal{G}$. In this case, the stack of branes α decays into a superposition of branes wrapping various different conjugacy classes. Which branes appear in the final configuration is determined by the Clebsch-Gordan multiplicities $\tilde{N}$ of the finite dimensional Lie algebra $\mathcal{G}$. More precisely, one finds

$$M\,(\alpha, \mathrm{id}) \;\longrightarrow\; \sum_{\gamma} \tilde{N}_{\sigma\alpha}^{\;\gamma}\; (\gamma, \mathrm{id})\,, \tag{3.4}$$

where $M = \dim(\sigma)$. The support for this statement comes from both the open string sector and the coupling to closed strings (see [16]).

A simple check of the rule (3.4) can be performed, if we extend the effective action for a stack of branes α by including non-constant gauge fields A_a. The extended action $\hat{S}$ has been derived in [16] and it is given by

$$\hat{S}_{M(\alpha,\mathrm{id})}(A_a) = S_{M\dim(\alpha)\,(0,\mathrm{id})}(Y_a + A_a)\,.$$

Here $A_a \in \mathrm{Mat}(M\dim(\alpha))$ and $Y_a \sim \mathbf{1}_M \otimes y_a$ involves the $\dim(\alpha)$-dimensional irreducible representation y_a of the Lie algebra $\mathcal{G}$. The construction of $\hat{S}$ is obviously consistent with the decay (3.4) which implies that $M\dim(\alpha)$ branes of the type $(0, \mathrm{id})$ can decay into M branes of type (α, id). Moreover, the formula for $\hat{S}$ has been derived within string perturbation theory in [16]. There $\hat{S}$ was identified as a special linear combination of Yang-Mills and Chern-Simons theory on a fuzzy sphere depending on α. Yang-Mills theories on fuzzy spheres are discussed in [41]–[43]. The Chern-Simons term is considered in [44]. For branes on S^3, the action $\hat{S}$ appears instead of the non-commutative Yang-Mills theory which was derived in [45] to describe the dynamics of branes in flat space with a non-vanishing B-field.

Irreducible M dimensional representations σ of $\mathcal{G}$ are still stationary points of this extended action $\hat{S}$. To test the rule (3.4) we study arbitrary fluctuations δA_a of the gauge field $A_a = \Lambda_a + \delta A_a \in \mathrm{Mat}(M\dim(\alpha))$ around a stationary point $\Lambda_a \sim \Lambda_a \otimes \mathbf{1}_{\dim(\alpha)}$. Here Λ_a are the representation matrices of the $M = \dim(\sigma)$-dimensional irreducible representation σ of $\mathcal{G}$. By construction, Λ_a and Y_a commute so that their sum $\Lambda_a + Y_a$ gives the tensor product representation $\sigma \times \alpha$ which decomposes into a sum of irreducibles γ with multiplicities $\tilde{N}_{\sigma\alpha}^{\ \gamma}$. Comparison with the construction of $\hat{S}$ shows that

$$\hat{S}_{M(\alpha,\mathrm{id})}(\Lambda_a + \delta A_a) = \sum_\gamma \hat{S}_{\tilde{N}_{\sigma\alpha}^{\ \gamma}\,(\gamma,\mathrm{id})}(\delta A_a) + \cdots ,$$

where we omitted terms involving massive fields that come with open strings stretching between different branes. The resulting action for the fluctuation field δA_a contains all the terms that are predicted by the rule (3.4).

Let us finally remark that the final configuration on the right hand side of (3.4) is only metastable. Whenever a superposition of branes appears in the final state one can find a renormalization group flow into another configuration of branes with lower mass. These flows, however, are generated by non-constant gauge fields A_a. For detailed explanations and computations the reader is referred to [16].

3.2 Condensation in the stringy regime

Now we would like to understand the dynamics of branes in the stringy regime. Proceeding along the lines of the previous subsection would force us to include all the higher order corrections to the effective action. Unfortunately, this problem is even more complicated than finding the non-abelian Born-Infeld action. Hence, we cannot hope to get a complete picture of the brane dynamics in the stringy regime.

But we could be somewhat less ambitious and ask whether the solutions we found in the large volume limit possess a deformation into the small volume theory and if so, which fixed points they correspond to. In this way we may overlook new stationary points of the stringy effective action that have no well behaved large k limit. On the other hand, the reduced program has a positive and very beautiful solution that is known from the work on the Kondo effect.

The Kondo model is designed to understand the effect of magnetic impurities on the low temperature conductance properties of a conductor. The latter may have electrons in several conduction bands. Let us say that there are k such bands. Now we can build several currents from the basic fermionic fields. Among them is the spin current $\vec{J}(y)$ which gives rise to a $\widehat{\mathcal{G}}_k$ current algebra. The coordinate y measures the radial distance from a spin s impurity at $y = 0$ to which the spin current couples. This coupling involves a $2s+1$-dimensional irreducible representation $\vec{\Lambda} = (\Lambda_a, a = 1, 2, 3)$ of $su(2)$ and it is of the form

$$H_{\mathrm{pert}} \ = \ \lambda \Lambda_a J^a(0)\,. \tag{3.5}$$

The operator H_{pert} acts on the tensor product $V^\sigma \otimes \mathcal{H}$ of the Hilbert space $\mathcal{H}$ for the unperturbed theory with the $2s + 1$ -dimensional quantum mechanical state space of our impurity. The formula (3.5) is simply the Hamiltonian formulation of the perturbations we would like to study, as one can see by comparison with formula (3.1) above.

Fortunately, a lot of techniques have been developed to deal with perturbations of the form (3.5). In fact, this problem is what Wilson's renormalization group techniques were designed for. From the old analysis we know that there are two different cases to be distinguished. When $2s > k$ ('under-screening') the low temperature fixed point of the Kondo model appears only at infinite values of λ. On the other hand, the fixed point is reached at a finite value $\lambda = \lambda^*$ of the renormalized coupling constant λ if $2s \leq k$ (exact- or over-screening resp.). In the latter case, the fixed points are described by non-trivial (interacting) conformal field theories. We can summarize the results on the spectrum of the fixed points by the formula [46]

$$\mathrm{tr}_{\,V^\sigma\otimes\mathcal{H}_j}\left(q^{H_0+H_{\rm pert}}\right)^{\rm ren}_{\lambda=\lambda^*} := \sum_l N_{\sigma j}^{\;\;l}\chi_l(q)\,. \tag{3.6}$$

Here, $H_0 = L_0 + c/24$ is the unperturbed hamiltonian, the superscript $^{\rm ren}$ stands for 'renormalized' and V^σ denotes the representation space of the representation σ of $su(2)$ or, more generally, of an arbitrary simple Lie algebra $\mathcal{G}$. The space $\mathcal{H}_j$ can be any of the $\widehat{\mathcal{G}}_k$-irreducible subspaces in the physical state space $\mathcal{H}$ of the theory. Formula (3.6) means that our perturbation with some irreducible representation σ interpolates continuously between a building block $\dim(\sigma)\chi_j(q)$ of the partition function of the UV-fixed point (i.e. $\lambda = 0$) and the sum of characters on the right hand side of the previous formula,

$$M\,\chi_j(q) \;\longrightarrow\; \sum_l N_{\sigma j}^{\;\;l}\chi_l(q)\,, \tag{3.7}$$

where $M = \dim(\sigma)$. We will now use this rule to find the spectra for the decay product of a stack of M branes Ξ. To this end, we need to start from the partition function describing open strings stretching between a stack of M branes of the type (α,ω) and a single brane (β,ω). This is given by M times the partition function (2.3), i.e. by a sum of characters with coefficients being integer multiples of M. We can now employ our rule (3.7) to determine the partition function of the system after perturbation with some irreducible M-dimensional representation σ of $\mathcal{G}$. The result is

$$\begin{aligned} Z^\omega_{M\alpha\,\beta}(q) := M Z^\omega_{\alpha\,\beta}(q) \;&\longrightarrow\; \sum_{j\in\mathcal{J}_k} n_{j\alpha}^{\omega;\beta}\sum_l N_{\sigma j}^{\;\;l}\chi_l(q) \\ &= \sum_{\gamma\in\mathcal{J}_k^\omega} n_{\sigma\alpha}^{\omega;\gamma}\, Z^\omega_{\gamma\,\beta}(q)\,. \end{aligned}$$

Here we used the property (2.4) of the coefficients n to express the right hand side as a linear combination of known partition functions. Since the coefficients on the right hand side are independent of β, we can summarize the result of our simple computation by the rule

$$M\,(\alpha,\omega) \;\longrightarrow\; \sum_{\gamma\in\mathcal{J}_k^\omega} n_{\sigma\alpha}^{\omega;\gamma}\,(\gamma,\omega) \tag{3.8}$$

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without any reference to the spectator brane (β, ω). Here, M denotes the dimension of the representation σ of the finite dimensional Lie algebra $\mathcal{G}$ and $\alpha \in \mathcal{J}_k^\omega$. This is the kind of result that we were looking for. Let us note that this reproduces the picture that we sketched at the end of the previous subsection when $\omega = \mathrm{id}$. In this case, the numbers n specialize to the fusion rules of $\widehat{\mathcal{G}}_k$ and they approach the fusion rules of the Lie algebra $\mathcal{G}$ when we send k to infinity so that we recover the formula (3.4). The formula (3.6) is the content of the 'absorption of the boundary spin'-principle [46] and it was previously applied to investigations of brane dynamics in [18, 47].

4. Conserved charges and twisted K-theory

We would like to see whether the described brane dynamics obey some conservation laws, i.e. if we can assign charges to the branes that are conserved in physical processes. So we are looking for some discrete abelian group $C(X)$, where X denotes the physical background, and a map from arbitrary brane configurations to $C(X)$ such that the map is invariant under renormalization group flows.

Let us denote the charge of a brane (α, ω) by $q_{(\alpha,\omega)} \in C(X)$. From the process (3.8) where a stack of $\dim(\sigma)$ branes of type (α, ω) condenses we get

$$\dim(\sigma)\, q_{(\alpha,\omega)} = \sum_{\gamma \in \mathcal{J}_k^\omega} n_{\sigma\alpha}^{\omega;\gamma}\, q_{(\gamma,\omega)}\,. \tag{4.1}$$

These equations express the condition for charge conservation under all the processes we have identified in the previous section. The requirement that eqs. (4.1) possess solutions places strong constraints on the group $C(\mathrm{SU}(N), K = k + N)$ of charges. We shall evaluate them completely for untwisted branes in the first subsection. Then we turn to the twisted branes which are more difficult to control. Nevertheless, we will obtain detailed information on $C(\mathrm{SU}(N), K)$. This is then summarized in the last subsection and compared to what is known on the twisted K-groups $K_H^*(\mathrm{SU}(N))$.

4.1 The charge of untwisted branes on $\mathrm{SU}(N)$

For branes wrapping ordinary conjugacy classes, i.e. $\omega = \mathrm{id}$, the integers n are given by the fusion rules N. The evaluation of eqs. (4.1) for $\alpha = 0$ leads to

$$q_\beta = \dim(\beta)\, q_0 \tag{4.2}$$

for all branes $\beta = (\beta, \mathrm{id})$ and with $q_0 = q_{(0,\mathrm{id})}$ being the charge of the point-like brane at the group unit e. Hence, all the brane charges of untwisted branes are integer multiples of q_0. If we normalize the charge of the point-like brane by $q_0 = 1$ we arrive at

$$q_\beta = \dim(\beta)\,. \tag{4.3}$$

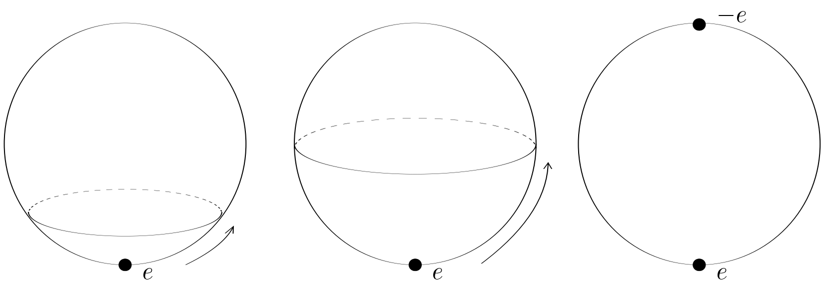

Figure 1: Brane dynamics on S^3: A stack of D0-branes at e can decay to a D2-brane. Putting more and more D0-branes at e the resulting brane will be localized further and further away from the group unit and eventually the decay product will be a single D0-brane at $-e$.

These equations form a subset of the equations (4.1). But we can see that the charges $q_\beta = \dim(\beta)$ solve the full set of eqs. (4.1) in the limit $k \to \infty$ where the N are just the Clebsch-Gordan multiplicities of the simple Lie algebra $su(N)$. In this limit, the equations express that the dimension of a tensor product of $su(N)$ representations is a sum of dimensions of its irreducible subrepresentations. For finite k, however, the fusion rules N differ from the Clebsch-Gordan multiplicities of $su(N)$ so that typically the right hand side of eqs. (4.1) with $q_\beta = \dim \beta$ is smaller than the left hand side. Hence, the equations can only hold if they are evaluated modulo some integer x that we need to determine. Charges then take values in the group $\mathbb{Z}_x$.

Let us first look at a simple example, $X = \mathrm{SU}(2)$. For level k the labels α lie in the range $0, \ldots, k$. We have a geometrical understanding of what the possible D-branes are. They are given by conjugacy classes which form 2-spheres embedded in $\mathrm{SU}(2) \cong S^3$. Their radius depends on α and for $\alpha = 0, k$ they degenerate to a point. $\alpha = 0$ describes a D0-brane at the origin e, $\alpha = k$ a D0-brane at $-e$.

Now consider a stack of D0-branes at e. This stack is expected to decay into a D2-brane on a 3-sphere with finite volume. If we put more and more D0-branes together the radius of the resulting D2-brane will first grow, then decrease, and finally a stack of $k+1$ D0-branes will decay to a D0-brane at $-e$ (see fig. 1). If we assign charge 1 to the D0-brane at e and want the charge to be conserved, the D0-brane at $-e$ must have charge $k+1$. On the other hand we could just translate the D0-brane from e to $-e$, and by taking orientation into account, this would lead to charge -1. Thus we have to identify $k+1$ and -1 which means that charge is only well-defined modulo $k+2$.

We can obtain the same result in a more algebraic way. To this end, we evaluate (4.1) for the simple current $\sigma = J = k$ and the fundamental 2-dimensional representation $\alpha = 1$. This gives

$$(k+1) \cdot 2 = \dim(k) \cdot 2 = q_{k-1} = k\,, \tag{4.4}$$

where we used that the product of the simple current with the fundamental representation of $\widehat{\mathcal{G}}_k$ gives the unique representation with label $\beta = k-1$ and $\dim(\beta) = k$. The equation (4.4) can only hold modulo $x = k+2$. One can show that this choice of x is consistent with all processes, i.e. that

$$\dim(\sigma)\dim(\alpha) = \sum_{\beta} N_{\sigma\alpha}^{\ \beta}\dim(\beta) \bmod (k+2)\,.$$

As we shall see later, it is always sufficient to evaluate the charge conservation condition only for simple currents and fundamental representations. The resulting restrictions are strong enough to guarantee charge conservation for all processes.

We are now turning to the more general case of $X = \mathrm{SU}(N)$. The task is to find the largest number x such that (4.1) is fulfilled modulo x. As we can generate all representations out of the fundamental ones, ω_i, $i = 1, \ldots, N-1$, we can reduce our problem to processes involving stacks of ω_i-branes. In other words, the general charge conservation condition is fulfilled if

$$\dim(\sigma)\, q_{\omega_i} = \sum_{\beta} N_{\sigma\omega_i}^{\ \beta}\, q_\beta \mod x \tag{4.5}$$

for all $i = 1, \ldots, N-1$ and $\sigma \in \mathcal{J}_k$. A rigorous prove of this statement can be found in appendix B.

Denote by $J = k\omega_1$ the generator of the simple current group $\mathbb{Z}_N$ of $\widehat{su}(N)_k$. It can be shown that it suffices to evaluate the equations (4.1) for stacks of $\dim(J)$ fundamental branes (see appendix B). Thus, the charge conservation condition reduces to

$$\dim(J) q_{\omega_i} = \sum_{\beta} N_{J\omega_i}^{\ \beta} q_\beta \bmod x \tag{4.6}$$

for all $i = 1, \ldots, N-1$. Taking the difference between both sides with $q_\alpha = \dim(\alpha)$ inserted, gives the following $N-1$ numbers a_i, (see (A.3))

$$a_i := \dim(J)\dim(\omega_i) - \sum_{\beta} N_{J\omega_i}^{\ \beta}\dim(\beta) = \frac{(k+1)\ldots\widehat{(k+i)}\ldots(k+N)}{(i-1)!\,(N-i)!}\,, \tag{4.7}$$

where the hat over a factor indicates that this factor is omitted. These numbers have to vanish modulo x. This means that x is given by the greatest common divisor of these numbers. It can be shown (see appendix C) that $x = \gcd(a_i)$ is given by

$$x = \frac{k+N}{\gcd(k+N, \mathrm{lcm}(1, \ldots, N-1))}\,. \tag{4.8}$$

Hence, the charge group of the untwisted branes for $X = \mathrm{SU}(N)$ is $\mathbb{Z}_x$ with x as in formula (4.8).

4.2 Charges of twisted branes on SU(N)

Let us now take a look at branes that wrap twisted conjugacy classes. As gluing automorphism we choose the reflection ω of the Dynkin diagram. Their action on the vertices of the Dynkin diagram induces the following map on the weight space,

$$\varpi(\lambda_1,\ldots,\lambda_{N-1}) = (\lambda_{N-1},\ldots,\lambda_1)\,, \tag{4.9}$$

where the λ_i are (finite) Dynkin labels. Details on our notations and some fundamental results on the representation theory of $su(N)$ can be found in appendix A.

As in the untwisted case we get a charge conservation condition,

$$\dim(\sigma)\, q_\alpha = \sum_{\beta\in\mathcal{J}_k^\omega} n_{\sigma\alpha}^{\omega;\beta}\, q_\beta \qquad \text{for all} \qquad \alpha\in\mathcal{J}_k^\omega\,. \tag{4.10}$$

We would like to perform a similar analysis as in the untwisted case but we are faced with the problem that the integers $n = n^\omega$ are a lot more difficult to handle than the fusion rules N. But even though we are not able to fully exploit these conditions, we can get some severe constraints on the charge group by symmetry considerations.

We will see that the numbers n are invariant under the action of some simple currents I,

$$n_{I\sigma\,\alpha}^{\ \beta} = n_{\sigma\,\alpha}^{\ \beta}\,. \tag{4.11}$$

To derive this result we look at the explicit expressions (2.5) for the numbers n and investigate what happens under the action of a simple current I:

$$\begin{aligned} n_{I\sigma\,\alpha}^{\ \beta} &= \sum_{\lambda\in\mathcal{J}_k^\omega} \frac{S^{\omega\,*}_{\lambda\beta} S^\omega_{\lambda\alpha} S_{\lambda\, I\sigma}}{S_{\lambda 0}} \\ &= \sum_{\lambda\in\mathcal{J}_k^\omega} e^{2\pi i Q_I(\lambda)}\, \frac{S^{\omega\,*}_{\lambda\beta} S^\omega_{\lambda\alpha} S_{\lambda\sigma}}{S_{\lambda 0}}\,. \end{aligned} \tag{4.12}$$

Here $Q_I(\lambda)$ is the monodromy charge of λ with respect to the simple current I. If it is zero, we infer that the coefficients n are invariant under the action of the simple current.

For a symmetric weight $\lambda = \varpi\lambda \in \mathcal{J}_k^\omega$ we know that

$$\varpi(J^i\,\lambda) \;=\; J^{N-i}\,\lambda\,. \tag{4.13}$$

This implies immediately that $Q_J(\lambda) = Q_{J^{N-1}}(\lambda)$. If N is odd, it follows that $Q_{J^i}(\lambda) = 0$ for all $i = 1,\ldots,N-1$. If N is even, we can only deduce that $Q_{J^i}(\lambda) = 0$ for i even. We thus arrive at the result that

$$n_{J^i\sigma\,\alpha}^{\ \beta} = n_{\sigma\,\alpha}^{\ \beta} \tag{4.14}$$

for arbitrary i if N is odd and for even i if N is even which is the precise formulation of the invariance properties of n we anticipated in eq. (4.11).

Assuming that there is at least one twisted brane which can be assigned a charge with value 1 we immediately deduce the following condition on the unknown integer x_ω

$$\dim(J^i\sigma) = \dim(\sigma) \mod x_\omega\,, \tag{4.15}$$

where the values for i depend on whether N is even or odd, as formulated before.

Let us first concentrate on the case that N is odd. Using (4.15) with $\sigma = 0, \omega_i$ we obtain

$$\dim(J) = 1 \qquad \mod x_\omega \tag{4.16}$$

$$\dim(J\omega_i) = \dim(\omega_i) \qquad \mod x_\omega\,. \tag{4.17}$$

The two relations combine into the following statement for the numbers a_i that were defined in eqs. (4.7) above,

$$a_i = \dim(J)\dim(\omega_i) - \dim(J\omega_i) = 0 \quad \mod x_\omega\,. \tag{4.18}$$

By definition, the greatest common divisor of these numbers a_i is x and hence we deduce that x_ω is a divisor of x, i.e. that the order of an element in the charge group for twisted branes cannot exceed the order of the charge subgroup from untwisted branes. It can be shown that $x_\omega = x$ does imply eqs. (4.15) but we cannot exclude that the eqs. (4.10) force x_ω to be smaller than x.

If N is even, we find x_ω not so strongly restricted by the eqs. (4.15). Introducing the integers $b_0 = \dim(J^2) - 1$ and $b_i = \dim(J^2\omega_i) - \dim(\omega_i)$ for $i = 1, \ldots, N-1$, one can show that x_ω must divide $\gcd(b_i)$. Note that $\gcd(b_i)$ is a possibly non-trivial integer multiple of x. For SU(4) we still get the result that x_ω divides x but already for SU(6) one finds situations where (4.15) can be fulfilled modulo $x_\omega > x$. There is some evidence that eqs. (4.15) provide enough restrictions for $N = 0 \bmod 4$ to guarantee that x_ω divides x.

4.3 Comparison with twisted K-theory

Before we explain what is known about the twisted K-groups $K^*_H(\mathrm{SU}(N))$, let us briefly summarize the results for $C(\mathrm{SU}(N), K)$ that we obtained in the previous two subsections. The charge group that governs the dynamics of branes in a $\widehat{su}(N)_k$ WZW-model is

$$C(\mathrm{SU}(N), K) = \mathbb{Z}_x \oplus \mathbb{Z}_y \oplus \cdots\,, \tag{4.19}$$

where x is given by (4.8) and y divides x_ω. In case $N \leq 5$ we know that x_ω (and hence y) must divide x and this remains true for $N > 5$ as long as N is odd. For even $N \geq 6$ we can only show that x_ω divides $\gcd(b_i)$ with the integers b_i being introduced in the last paragraph of the previous subsection. In general, $\gcd(b_i)$ could be some possibly non-trivial integer multiple of x but it is very likely that $\gcd(b_i) = x$ when $N = 0 \bmod 4$. Furthermore, one can show that x and y are equal for $N = 3$.

The two types of branes we have been looking at preserve the full chiral symmetry of the WZW model. At the moment, these are the only types of branes that can be treated within boundary conformal field theory. It is possible, however, that other types of stable branes exist that could give rise to additional contributions in the group $C(\mathrm{SU}(N), K)$ of brane charges, even though this is unlikely to be the case for $N = 2, 3$. This uncertainty is reflected by the $\oplus \cdots$ in eq. (4.19).

According to the proposal of Bouwknegt and Mathai, our results on the group $C(\mathrm{SU}(N), K)$ should be compared to the twisted K-groups $K_H^*(\mathrm{SU}(N))$. Unfortunately, the latter have not been computed yet.

The definition of $K_H^*(X)$ uses the space of sections in a bundle over X with fiber being the algebra of compact operators on a separable Hilbert space. This space of sections can be turned into an algebra and it is known that algebras of this form are classified by elements of $H^3(X, \mathbb{Z})$. In other words, there exists some way of assigning an algebra $\mathcal{A}_H$ to any choice of $H \in H^3(X, \mathbb{Z})$. The K-group of this algebra is denoted by $K_H^*(X)$. If H vanishes, the algebra $\mathcal{A}_H$ factorizes globally into functions on X and compact operators. Hence, by Morita invariance of K-theory, $K_{H=0}^*(X)$ coincide with ordinary K-groups.

One way to calculate such K-groups makes use of Atiyah-Hirzebruch spectral sequences.[4] These start from the de Rham cohomology groups and then proceed through a sequence of complexes whose cohomology stabilizes after a finite number of steps. The resulting cohomology provides some information on the desired K-group, though there is still some extension problem to solve. Generically, the latter may have several solutions. In any case, the problem of these computations for $K_H^*(G)$ starts earlier because almost nothing is known about the differentials that appear in the sequence of complexes. Only for the first non-trivial step, the required differential was obtained by Rosenberg in [48]. This suffices to compute the twisted K-group for $G = \mathrm{SU}(2)$. The result is

$$K_H^*(\mathrm{SU}(2)) = \mathbb{Z}_K \,.$$

Here $H = K\Omega_3$ and Ω_3 is the normalized volume form of the unit sphere. For $G = \mathrm{SU}(3)$, Rosenberg's results still allow to show that

$$K_H^*(\mathrm{SU}(3)) = \mathbb{Z}_r + \mathbb{Z}_r \,,$$

where r is known to divide K. If all the higher differentials that are not determined by the result of Rosenberg would vanish, then one would get $r = K$. Hence, the comparison with our CFT results suggests that the higher differentials do not vanish, at least for even $K = k + 3$.

[4]We are very grateful to I. Brunner and G. Moore for their explanations on AHSS in the context of twisted K-theory.

It would be highly desirable to get more results on the twisted K-groups. At the moment, the restrictions on $C(\mathrm{SU}(N), K)$ that we obtained by studying renormalization group flows in WZW-models provide highly non-trivial predictions for $K^*_H(\mathrm{SU}(N))$. It was suggested to us by Wassermann that the techniques in [49, 50] could lead to a computation of $K^*_H(\mathrm{SU}(N))$ which would employ the results of [51] on equivariant twisted K-theory.

5. Conclusions and open problems

In this work we studied brane dynamics on $\mathrm{SU}(N)$ and formulated conservation laws for these dynamics. The conserved charges take values in some finite abelian group $C(\mathrm{SU}(N), K)$. While we were not able to determine $C(\mathrm{SU}(N), K)$ completely, we obtained a number of strong restrictions on its structure. These are reflected in our formula (4.19). Our results are consistent with the proposal $C(\mathrm{SU}(N), K) = K^*_H(\mathrm{SU}(N))$ of [10], but since so little is known about K^*_H the comparison was restricted to $N = 2, 3$.

Some difficulties in our analysis of $C(\mathrm{SU}(N), K)$ were related with branes wrapping twisted conjugacy classes. A better understanding of the numbers n^ω that determine the partition functions for such theories would certainly lead to a more precise prediction of y, similar to what we achieved in the case of untwisted branes. This applies, in particular, to the study of $\mathrm{SU}(N)$ with N even. It would be interesting to re-interprete the numbers $n^{\omega,\beta}_{\sigma\alpha}$ for finite α, β within the framework of non-commutative geometry. This is possible for $\omega = \mathrm{id}$ and in this case it leads to fuzzy geometries. A similar interpretation for branes wrapping twisted conjugacy classes does not exist.

An extension of our discussion to other groups G is possible. Most of the basic ideas we have used do not depend on the specific choice $G = \mathrm{SU}(N)$. Only in our evaluation of the charge conservation condition (4.1) in section 4 we exploited some simplifying features that hold for $G = \mathrm{SU}(N)$.

It would also be interesting to go beyond these examples and to understand the importance of twisted K-theory for branes in curved backgrounds more generally from the nature of the fields that condense upon bound state formation. Such arguments would be analogous to the relation between ordinary K-theory and tachyon condensation (see e.g. [3, 52] and references therein). Note, however, that in the cases we studied above the dynamics is driven by massless fields rather than conventional tachyons. Let us also stress that all the processes we have considered involve finite stacks of branes. This is in some contrast to the construction of twisted K-theory for non-torsion H-fields which involves taking some limit $M \to \infty$ [10]. This has motivated the authors of [10] to speculate about some relation with processes on an infinite stack of branes.

Finally, we would like to mention that many of the results on branes in WZW-models descend to other models of conformal field theory through orbifold and coset constructions. Thereby, our results could be used to extend the investigations in [53] and they should bear some relevance even for the behavior of branes in Gepner models which describe strings and branes on certain Calabi-Yau spaces deep in the stringy regime. We plan to return to these issues in a forthcoming publication.

Acknowledgments

We would like to thank A.Yu. Alekseev, V. Braun, A. Carey, C.S. Chu, D. Freed, R. Helling, V. Mathai, A. Recknagel, T. Quella, A. Wassermann and in particular I. Brunner and G. Moore for stimulating and very useful discussions. This research was carried out in part for the Clay Mathematics Institute.

A. Some representation theory

In this appendix we will briefly review some facts in $\widehat{su}(N)$-representation theory. Details can be found e.g. in [54].

An affine weight λ can be expanded in fundamental weights,

$$\lambda = \lambda_0\omega^0 + \lambda_1\omega^1 + \cdots + \lambda_{N-1}\omega^{N-1} .$$

The expansion coefficients are the Dynkin labels. When we consider representations at level k, the zeroth Dynkin label is fixed by the others,

$$\lambda_0 = k - \sum_{i=1}^{N-1} \lambda_i ,$$

therefore λ is determined by its finite Dynkin labels $(\lambda_1, \ldots, \lambda_{N-1})$.

The fundamental weights are then given by

$$\omega_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) ,$$

the vacuum representation is $(0, \ldots, 0)$.

We are interested in integrable highest-weight representations. We find that their highest weight λ has to be dominant, i.e. the Dynkin labels of λ have to be non negative integers. For a given level k there are only finitely many dominant weights restricted by

$$\sum_{i=1}^{N-1} \lambda_i \leq k .$$

Instead of using Dynkin labels we can specify a weight λ in terms of its partition

$$\lambda = \{\ell_1; \ell_2; \ldots; \ell_{N-1}\}$$

where

$$\ell_i = \lambda_i + \cdots + \lambda_{N-1}\,.$$

The dimension of the representation of the finite simple Lie algebra $su(N)$ belonging to the highest weight λ can be easily given by the partition,

$$\dim(\lambda) = \prod_{1\le i<j\le N} \frac{\ell_i - \ell_j + j - i}{j-i}\,, \tag{A.1}$$

where $\ell_N = 0$.

Partitions are also useful in calculating Clebsch-Gordan coefficients via the Littlewood-Richardson rule (see e.g. [54]). The fusion rules of the affine Lie algebra can be obtained from the Clebsch-Gordan coefficients by a suitable truncation.

Let us consider an example that we will need for our discussion. We consider the fusion of the simple current generator $J = (k, 0, \ldots, 0)$ with a fundamental weight ω_i. In the tensor product decomposition we find two representations, $J + \omega_i$ and $J - \omega_1 + \omega_{i+1}$ (setting $\omega_N = 0$). The first one has $\ell_1 = k+1$ and is ignored because of the truncation at level k, the second one remains.

The dimensions of the corresponding representations of the finite Lie algebra fulfil

$$\dim(J)\dim(\omega_i) \;=\; \sum_\beta \tilde{N}_{J\omega_i}^{\beta} \dim(\beta)\,, \tag{A.2}$$

where $\tilde{N}$ denote the finite tensor-product coefficients.

When we substitute $\tilde{N}$ by the fusion rules N of the affine Lie algebra, this equation is not longer valid. In our example, the difference between both sides is then given by $\dim(J+\omega_i)$ which is (using (A.1))

$$a_i := \dim(J+\omega_i) = \frac{(k+1)\cdots\widehat{(k+i)}\cdots(k+N)}{(i-1)!\,(N-i)!}\,. \tag{A.3}$$

B. Some lemmas used in section 4

We consider the affine Lie algebra $\widehat{su}(N)_k$. We denote the fundamental weights by ω_i, $i = 1, \ldots, N-1$. By $q_\alpha = \dim(\alpha)$ we denote the dimension of the irreducible highest-weight representation of the horizontal subalgebra corresponding to α.

Lemma 1. *Suppose*

$$q_\sigma\, q_{\omega_i} = \sum_\beta N_{\sigma\omega_i}^{\;\beta}\; q_\beta \mod x \quad \forall\; i, \sigma\,.$$

Then

$$q_\sigma\, q_\alpha = \sum_\beta N_{\sigma\alpha}^{\;\beta}\; q_\beta \mod x \qquad \forall\; \alpha, \sigma\,.$$

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Proof. We will proof the lemma by induction over the sum of the finite Dynkin labels $\ell_1(\alpha) = \sum_{i=1}^{N-1} \alpha_i$. The equation obviously holds for $\ell_1(\alpha) = 0$ and for $\ell_1(\alpha) = 1$ (fundamental weights).

Suppose now that the assertion is valid for labels with $\ell_1 \leq \ell$. For a label α with $l_1(\alpha) \leq l+1$ we denote by $i = i(\alpha)$ the number between 0 and $N-1$ satisfying $\ell_j(\alpha) = \ell + 1$ for $1 \leq j \leq i$ and $\ell_j(\alpha) \leq \ell$ for $j > i$. Clearly the equation holds for weights satisfying $i = 0$. By induction we show that it holds for all i and therefore for all weights with $\ell_1 \leq \ell + 1$.

Let α be a weight with $\ell_1(\alpha) = \ell + 1$. Then this weight appears once in the fusion of the weight $\alpha' = \alpha - \omega_{i(\alpha)}$ with $\ell_1(\alpha') = \ell$ and the fundamental weight $\omega_{i(\alpha)}$. The other weights λ appearing in the fusion have $i(\lambda) < i(\alpha)$. Assuming that the equation is valid for these λ and using the associativity of the fusion product we show that the equation holds for α,

$$
\begin{aligned}
\sum_\beta N_{\sigma\alpha}^{\ \beta}\, q_\beta \quad &= \quad \sum_\beta N_{\alpha'\omega_i}^{\ \alpha}\, N_{\sigma\alpha}^{\ \beta}\, q_\beta \\
&= \quad \sum_\beta \Big[\sum_\lambda N_{\alpha'\omega_i}^{\ \lambda}\, N_{\sigma\lambda}^{\ \beta} - \sum_{\lambda\neq\alpha} N_{\alpha'\omega_i}^{\ \lambda}\, N_{\sigma\lambda}^{\ \beta} \Big]\, q_\beta \\
&= \quad \sum_\beta \Big[\sum_\lambda N_{\sigma\omega_i}^{\ \lambda}\, N_{\alpha'\lambda}^{\ \beta} - \sum_{\lambda\neq\alpha} N_{\alpha'\omega_i}^{\ \lambda}\, N_{\sigma\lambda}^{\ \beta} \Big]\, q_\beta \\
&\overset{\bmod x}{=} \quad \sum_\lambda N_{\sigma\omega_i}^{\ \lambda}\, q_{\alpha'}\, q_\lambda - \sum_{\lambda\neq\alpha} N_{\alpha'\omega_i}^{\ \lambda}\, q_\sigma\, q_\lambda \\
&\overset{\bmod x}{=} \quad q_\sigma\, q_{\omega_i}\, q_{\alpha'} - q_{\alpha'}\, q_{\omega_i}\, q_\sigma + N_{\alpha'\omega_i}^{\ \alpha}\, q_\sigma\, q_\alpha \\
&= \quad q_\sigma\, q_\alpha\,.
\end{aligned} \tag{B.1}
$$

This completes the proof of lemma 1.

We will show in the following that it is sufficient to evaluate the charge conservation condition for fundamental representations ω_i and the simple current generator J.

Lemma 2. *Suppose*

$$
q_J\, q_{\omega_i} = \sum_\beta N_{J\omega_i}^{\ \beta}\, q_\beta \mod x \quad \forall\ i\,. \tag{B.2}
$$

Then

$$
q_\sigma\, q_{\omega_i} = \sum_\beta N_{\sigma\omega_i}^{\ \beta}\, q_\beta \mod x \quad \forall\ i, \sigma\,. \tag{B.3}
$$

Proof. Let us first remark that the equation certainly holds for $\ell_1(\sigma) < k$, because then the fusion matrices N coincide with the finite tensor-product coefficients. We are now going to proof the statement:

For all $i = 1, \ldots, N-1$ and $\ell = 0, \ldots, k-1$ the following is true:

A.

$$\sum_\beta N_{\sigma\omega_j}^{\ \beta}\ q_\beta \ \stackrel{\text{mod } x}{=}\ q_\sigma\, q_{\omega_j} \qquad \begin{array}{r} \forall\ j = 1, \ldots, N-1 \qquad\qquad\qquad\qquad\qquad\quad \\ \forall\ \sigma \ \text{ with } \ell_1(\sigma) = k\ ,\ \ell_j(\sigma) \leq \ell + 1 \text{ for } j \geq 2 \\ \ell_j(\alpha) \leq \ell \text{ for } j \geq i+1 \end{array} \tag{B.4}$$

B.

$$\dim(\beta) \stackrel{\text{mod } x}{=} 0 \qquad \forall\ \beta \,\text{with} \quad \begin{array}{r} \ell_1(\beta) = k+1\ ,\ \ell_j(\beta) \leq \ell + 2 \text{ for } j \geq 2 \\ \ell_j(\beta) \leq \ell + 1 \text{ for } j \geq i+1 \quad . \end{array} \tag{B.5}$$

We proof this proposition by induction over ℓ and i. We start with $\ell = 0, i = 1$. Part A is fulfilled because of (B.2). For part B consider a weight β with $\ell_1(\beta) = k+1, \ell_2(\beta) \leq 1$. Then $\beta = J + \omega_j$ for some j. This is just the truncated weight in the fusion of J and ω_j, therefore

$$\dim(J + \omega_j) = q_{\omega_j}\, q_J - \sum_\beta\ N_{\omega_j J}^{\ \beta}\ q_\beta \ \stackrel{\text{mod } x}{=}\ 0\,.$$

We note that the statements A and B for $\ell, i = N-1$ are equivalent to the statements for $\ell + 1, i = 1$. For the induction process we only have to show the step $(\ell, i) \Rightarrow (\ell, i+1)$.

Assume that $A_{\ell,i}$ and $B_{\ell,i}$ are valid. Let α be a label with $\ell_1(\alpha) = k, \ell_2(\alpha) \leq \ell + 1$. The fusion of α and ω_i differs from the finite tensor-product decomposition just by representations β with $\ell_1(\beta) = k+1, \ell_2(\beta) \leq \ell + 2$ and $\ell_{i+1}(\beta) \leq \ell + 1$. From $B_{\ell,i}$ we know that their dimensions vanish modulo x and hence

$$\sum_\beta N_{\alpha\omega_i}^{\ \beta}\, q_\beta \ \stackrel{\text{mod } x}{=}\ q_\alpha\, q_{\omega_i} \quad \text{for } \ell_1(\alpha) = k\,, \quad \ell_2(\alpha) \leq \ell + 1\,. \tag{B.6}$$

Now we will proof $A_{\ell,i+1}$. Let α be a label with $\ell_1(\alpha) = k$ and $\ell_2(\alpha) = \cdots = \ell_{i+1}(\alpha) = \ell + 1$, $\ell_j(\alpha) \leq \ell$ for $j \geq i+2$. We then define

$$\alpha' \ = \ \{k; \underbrace{\ell; \ldots; \ell}_{i\text{–times}}; \ell_{i+2}; \ldots\}\,.$$

α occurs once in the fusion of α' and ω_i, all the other labels occurring in the fusion fulfil the requirements of $A_{\ell,i}$. Hence

$$\begin{aligned}
\sum_\beta N_{\alpha\omega_j}^{\ \ \beta}\, q_\beta = & \sum_\beta N_{\alpha'\omega_i}^{\ \ \alpha}\, N_{\alpha\omega_j}^{\ \ \beta}\, q_\beta \\
= & \sum_\beta \left[\sum_\lambda N_{\alpha'\omega_i}^{\ \ \lambda}\, N_{\lambda\omega_j}^{\ \ \beta} - \sum_{\lambda\neq\alpha} N_{\alpha'\omega_i}^{\ \ \lambda}\, N_{\lambda\omega_j}^{\ \ \beta} \right] q_\beta \\
\overset{\mathrm{mod}\ x}{=} & \sum_{\lambda,\beta} N_{\alpha'\omega_j}^{\ \ \lambda}\, N_{\lambda\omega_i}^{\ \ \beta}\, q_\beta - \sum_{\lambda\neq\alpha} N_{\alpha'\omega_i}^{\ \ \lambda}\, q_\lambda\, q_{\omega_j} \\
\overset{\mathrm{mod}\ x}{=} & \sum_\lambda N_{\alpha'\omega_j}^{\ \ \lambda}\, q_\lambda\, q_{\omega_i} - \sum_\lambda N_{\alpha'\omega_i}^{\ \ \lambda}\, q_{\omega_j}\, q_\lambda + N_{\alpha'\omega_i}^{\ \ \alpha}\, q_\alpha\, q_{\omega_j} \\
\overset{\mathrm{mod}\ x}{=} & q_{\alpha'}\, q_{\omega_j}\, q_{\omega_i} - q_{\alpha'}\, q_{\omega_i}\, q_{\omega_j} + q_\alpha\, q_{\omega_j} \\
= & q_\alpha\, q_{\omega_j}\,.
\end{aligned} \tag{B.7}$$

Now we have to show $B_{\ell,i+1}$. Let β_0 be a label of the form

$$\beta_0 \ = \ \{k+1; \underbrace{\ell+2;\ldots;\ell+2}_{i\text{-times}}; \underbrace{\ell+1;\ldots;\ell+1}_{(j-i-1)\text{-times}}; \ell_{j+1}(\beta_0);\ldots;\ell_{N-1}(\beta_0)\}$$

with $\ell_{j+1}(\beta_0) \leq \ell$ and define

$$\beta' = \{k; \underbrace{\ell+1;\ldots;\ell+1}_{i\text{-times}}; \underbrace{\ell;\ldots;\ell}_{(j-i-1)\text{-times}}; \ell_{j+1}(\beta_0);\ldots;\ell_{N-1}(\beta_0)\}\,.$$

Then β_0 appears once in the finite tensor product of β' and ω_j. It belongs to the representations that are truncated by going over to the fusion rules of the affine Lie algebra. For the other truncated representations β we know from $B_{\ell,i}$ that $\dim(\beta) = 0 \bmod x$. But since $A_{\ell,i+1}$ is applicable to $\alpha = \beta'$ we get $\dim(\beta_0) = 0 \bmod x$. This completes the proof.

C. Evaluation of $\gcd(a_i)$

Lemma 3. *Let the numbers a_i be defined as in (4.7). Then their greatest common divisor is given by*

$$x := \gcd(a_i) \ = \ \frac{k+N}{\gcd(k+N, \mathrm{lcm}(1,\ldots,N-1))}\,.$$

Proof. We are only going to give a sketch of the proof. Let us rewrite the numbers a_i by introducing

$$b(N-1) \ = \ \frac{(N-1)!}{\mathrm{lcm}(1,\ldots,N-1)}$$

as

$$a_i = \frac{(k+1)\cdots\widehat{(k+i)}\cdots(k+N-1)}{b(N-1)} \binom{N-1}{i-1} \frac{k+N}{\mathrm{lcm}(1,\ldots,N-1)}\,.$$

An important observation is that the first factor in a_i is always an integer. As also the binomial coefficient is an integer, we can see that x is a divisor of all a_i.

It remains to show that it is already the greatest common divisor. Let p be a prime number. We determine the maximum y and the corresponding i such that $p^y|(k+i)$. Then one can show that $p \not| (a_i/x)$.

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